ON THE NONSTATIONARY MOTION OF A WING WITH RECTANGULAR PLANFORM

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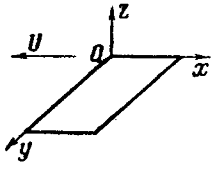
We demonstrate some results in the investigation of the unsteady motion of a thin rigid wing of finite aspect ratio and rectangular planform in supersonic flow of arbitrary velocity variation; this includes passage through a gust or a shock front. The problem is linearized. In the first part of the paper one finds the solution of the problem for the case of a change in wing angle of attack according to the law $e^{at}(-\infty < t < 0)$; in the second part, the obtained particular solution is used to examine cases where the angle of attack of the wing changes arbitrarily with time. Problems of this sort were examined by Krassitshchikova [1]. In the present paper, a closed form solution is obtained for the case of a wing of rectangular planform with account of edge effects.

1. We shall consider the straight-line forward motion of a thin, rigid wing of finite span and rectangular planform, moving in an infinite region of fluid and at rest at infinity. Superimposed on this basic motion, with constant supersonic velocity U, are additional small nonstationary motions.

We shall investigate the perturbed motion in a moving system of coordinates fixed to the wing and moving with the velocity U. The x-axis is in the direction opposite to the motion, the y-axis is in the spanwise direction, and the z-axis is upward (see figure).



We shall assume that the nonstationary motion of the wing produces small disturbances in the flow and that the perturbed flow has a potential. Then, as is known, the perturbation velocity potential $\phi(x, y, z, t)$ satisfies a linear differential equation which, in the moving system of coordinates, has the form:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \frac{1}{a^2}\left(\frac{\partial^2\varphi}{\partial t^2} + 2U\frac{\partial^2\varphi}{\partial x\partial t} + U^2\frac{\partial^2\varphi}{\partial x^2}\right) \qquad (1.1)$$

where $a = \sqrt{(dp/d\rho)}$ is the speed of sound in the undisturbed fluid.

The field is disturbed in that part of the field which is bounded by the envelope of Mach cones with vertices on the wing contour. Outside this region the velocity potential and its derivatives are equal to zero:

$$\phi = 0 \tag{1.2}$$

On the wing surface L the boundary condition is satisfied:

$$\frac{\partial\varphi}{\partial z} = f(t) \qquad \text{for } z = 0 \tag{1.3}$$

where $f(t)$ is an arbitrary function of its argument which is given on the semi-infinite interval $(-\infty, 0)$, with a finite number of points of discontinuity of first order, and sufficiently smooth at $-\infty$.

Everywhere in the x, y plane where the fluid is disturbed, but outside the plane of the wing and the vortex sheet,

$$\phi = 0 \tag{1.4}$$

The potential ϕ is an odd function with respect to the z-coordinate, $\phi(x, y, -z, t) = \phi(x, y, z, t)$; therefore the solution of the problem need be investigated in only the upper half-region.

Thus, it is necessary to determine a function $\phi(x, y, z, t)$ which satisfies equation (1.1), conditions (1.2), (1.3), (1.4) and is equal to zero at infinity, together with its derivatives.

The pressure on the wing is determined from the equation

$$p(x, y, 0, t) = p^+ - p^- = 2\rho_\infty\left[\frac{\partial\varphi(x, y, 0, t)}{\partial t} + U\frac{\partial\varphi(x, y, 0, t)}{\partial x}\right] \tag{1.5}$$

2. We shall find a particular solution of equation (1.1) for the case where the velocity component normal to the wing changes with time according to the relation

$$\left[\frac{\partial\varphi}{\partial z}\right]_{z=0} = e^{\alpha t} \qquad (-\infty \leqslant t \leqslant 0,\ \alpha > 0) \tag{2.1}$$

Letting $M = U/a$ denote the Mach number of the basic flow, we rewrite equation (1.1) in the following form:

$$-(M^2-1)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} - 2\frac{M}{a}\frac{\partial^2\varphi}{\partial x\partial t} = 0 \tag{2.2}$$

We shall look for a perturbation velocity potential $\phi(x, y, z, t)$ which is a solution of equation (2.1), in the form

$$\varphi(x, y, z, t) = e^{\alpha t+\beta x}\psi(x, y, z) \tag{2.3}$$

For the function $\psi(x, y, z)$ we obtain the equation

$$-(M^2-1)\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\left[-2(M^2-1)\beta-2M\frac{\alpha}{a}\right]\frac{\partial\psi}{\partial x}+$$
$$+\left[-(M^2-1)\beta^2-2M\frac{\alpha}{a}\beta-\frac{\alpha^2}{a^2}\right]\psi=0 \tag{2.4}$$

Making use of the arbitrary β, we require that

$$-2(M^2-1)\beta-2M\frac{\alpha}{a}=0, \quad \text{or} \quad \beta=-\frac{M}{M^2-1}\frac{\alpha}{a} \tag{2.5}$$

Equation (2.4) is transformed to

$$\frac{1}{M^2-1}\left(\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)-\frac{\partial^2\psi}{\partial x^2}+\lambda^2\psi=0 \qquad \left(\lambda^2=\frac{\alpha^2}{a^2}\frac{1}{(M^2-1)^2}\right) \tag{2.6}$$

Making the usual change of variables,

$$x_1=\lambda x, \quad y_1=\lambda y\sqrt{M^2-1}, \quad z_1=\lambda z\sqrt{M^2-1} \tag{2.7}$$

equation (2.6) takes the form

$$\frac{\partial^2\psi}{\partial y_1^2}+\frac{\partial^2\psi}{\partial z_1^2}-\frac{\partial^2\psi}{\partial x_1^2}+\psi=0 \tag{2.8}$$

On the basis of (1.2), (1.3) and (1.4) we obtain the following conditions for $\psi(x_1, y_1, z_1)$:

in the undisturbed field, the function ψ and its derivatives are zero:

$$\psi=0 \tag{2.9}$$

on the wing surface L_1

$$\frac{\partial\psi}{\partial z_1}=\frac{1}{\lambda\sqrt{M^2-1}}e^{-\nu x_1} \quad \text{for } z_1=0 \qquad \left(\nu=\frac{\beta}{\lambda}\right) \tag{2.10}$$

in the plane $z_1 = 0$, outside the plane of the wing and the vortex sheet,

$$\psi = 0 \tag{2.11}$$

Let us consider an auxiliary problem. Let $\psi^*(x_1, y_1, z_1)$ satisfy the equation

$$\frac{\partial^2\psi^*}{\partial y_1^2}+\frac{\partial^2\psi^*}{\partial z_1^2}-\frac{\partial^2\psi^*}{\partial x_1^2}=0 \tag{2.12}$$

and the conditions

$$\frac{\partial\psi^*}{\partial z_1}=f_1(x_1) \quad \text{on } L_1 \text{ for } z_1=0 \tag{2.13}$$

and conditions analogous to (2.9) and (2.11) outside L_1.

To investigate this auxiliary problem, we shall start from the expression for a function $\psi_0^*(x_1, y_1, z_1)$ which, like the function $\psi^*(x_1, y_1, z_1)$ satisfies the differential equation (2.12) and conditions analogous to (2.9) and (2.11) outside L_1; however, the condition (2.13) on L_1 for $z_1 = 0$ will be

$$\frac{\partial \psi_0^*}{\partial z_1} = 1 \tag{2.14}$$

The expression for the function $\psi_0^*(x_1, y_1, z_1)$ was given by Busemann [2]. The value of its derivative with respect to x_1, for $z_1 = 0$, is

$$\left[\frac{\partial \psi_0^*(x_1, y_1, z_1)}{\partial x_1}\right]_{z_1=0} = \begin{cases} -\pi^{-1} \arccos(1 - 2y_1/x_1) & \text{for } x_1 > y_1 \\ -1 & \text{for } x_1 < y_1 \end{cases} \tag{2.15}$$

A relation has been established between the functions $\psi_0^*(x_1, y_1, z_1)$ and $\psi^*(x_1, y_1, z_1)$ [3].

$$(\psi^* \; x_1, y_1, z_1) = \int_{A_1}^{A} \left[\frac{\partial \psi^*[(x_1 - \xi_1), y_1, z_1]}{\partial z_1}\right]_{z_1=0} \frac{\partial \psi_0^*(\xi_1, y_1, z_1)}{\partial \xi_1} d\xi_1 =$$

$$= \int_{A_1}^{A} f_1(x_1 - \xi_1) \frac{\partial \psi_0^*(\xi_1, y_1, z_1)}{\partial \xi_1} d\xi_1$$

for an arbitrary point $A(x_1, y_1, z_1)$ of the disturbed region; here the point A lies on the envelope of Mach cones.

For points on the wing,

$$\psi^*(x_1, y_1, 0) = \int_0^{x_1} \left[\frac{\partial \psi^*[(x_1 - \xi_1), y_1, z_1]}{\partial z_1}\right]_{z_1=0} \left[\frac{\partial \psi_0^*(\xi_1, y_1, z_1)}{\partial \xi_1}\right]_{z_1=0} d\xi_1 =$$

$$= \int_0^{x_1} f_1(x_1 - \xi_1) \left[\frac{\partial \psi_0^*(\xi_1, y_1, z_1)}{\partial \xi_1}\right]_{z_1=0} d\xi_1 \tag{2.16}$$

The expressions (2.16) make it possible to find a connection between the functions $\psi^*(x_1, y_1, z_1)$ and $\psi(x_1, y_1, z_1)$ in the plane $z_1 = 0$, using operational methods. Let

$$F(p, y_1, z_1) = p \int_0^{\infty} e^{-px_1} \psi(x_1, y_1, z_1)\, dx_1$$

$$F^*(p, y_1, z_1) = p \int_0^{\infty} e^{-px_1} \psi^*(x_1, y_1, z_1)\, dx_1 \tag{2.17}$$

be the transforms of the functions $\psi(x_1, y_1, z_1)$ and $\psi^*(x_1, y_1, z_1)$.

The functions $F(p, y_1, z_1)$ and $F^*(p, y_1, z_1)$ satisfy the partial differential equations

$$\frac{\partial^2 F}{\partial y_1^2} + \frac{\partial^2 F}{\partial z_1^2} - (p^2 - 1) F = 0, \qquad \frac{\partial^2 F^*}{\partial y_1^2} + \frac{\partial^2 F^*}{\partial z_1^2} - p^2 F^* = 0 \tag{2.18}$$

Evidently it is possible to write the following relation between $F(p, y_1, z_1)$ and $F^*(p, y_1, z_1)$:

$$F(y_1, z_1, p) = F^*(y_1, z_1, \sqrt{p^2-1}), \quad F(y_1, 0, p) = F^*(y_1, 0, \sqrt{p^2-1}) \tag{2.19}$$

We shall use the expression (2.16) for the function $\psi^*(x_1, y_1, 0)$:

$$\psi^*(x_1, y_1, 0) = \int_0^{x_1} \left[\frac{\partial\psi^*[(x_1-\xi_1), y_1, z_1]}{\partial z_1}\right]_{z_1=0} \left[\frac{\partial\overset{*}{\psi_0}(\xi_1, y_1, z_1)}{\partial\xi_1}\right]_{z_1=0} d\xi_1$$

we define

$$T(p, y_1, 0) = p\int_0^\infty e^{-px_1}\left[\frac{\partial\psi_0^*(x_1, y_1, z_1)}{\partial x_1}\right]_{z_1=0} dx_1$$

$$\left[\frac{\partial F^*(p, y_1, z_1)}{\partial z_1}\right]_{z_1=0} = p\int_0^\infty e^{-px_1}\left[\frac{\partial\psi^*(x_1, y_1, z_1)}{\partial z_1}\right]_{z_1=0} dx_1 \tag{2.20}$$

From the inversion theorem, we obtain for $F^*(y_1, 0, p)$:

$$F^*(y_1, 0, p) = \frac{1}{p}\left[\frac{\partial F^*(y_1, z_1, p)}{\partial z_1}\right]_{z_1=0} T(y_1, 0, p)$$

Making use of (2.19) it is possible to obtain an expression for $F(y_1, 0, p)$, namely

$$F(y_1, 0, p) = F^*(y_1, 0, \sqrt{p^2-1}) =$$

$$= \frac{1}{\sqrt{p^2-1}}\left[\frac{\partial F^*(y_1, z_1, \sqrt{p^2-1})}{\partial z_1}\right]_{z_1=0} T(y_1, 0, \sqrt{p^2-1})$$

From (2.19)

$$\left[\frac{\partial F^*(y_1, z_1, \sqrt{p^2-1})}{\partial z_1}\right]_{z_1=0} = \left[\frac{\partial F(y_1, z_1, p)}{\partial z_1}\right]_{z_1=0}$$

Therefore

$$F(p, y_1, \theta) = \frac{1}{\sqrt{p^2-1}}\left[\frac{\partial F(y_1, z_1, p)}{\partial z_1}\right]_{z_1=0} T(y_1, 0, \sqrt{p^2-1})$$

Due to (2.10), $\left[\dfrac{\partial F(y_1, z_1, p)}{\partial z_1}\right]_{z_1=0} = \dfrac{1}{\lambda\sqrt{M^2-1}}\dfrac{p}{p+\nu}$

and thus the expression for representing the required function $\psi(x_1, y_1, 0)$ finally has the following form:

$$F(p,\ y_1,\ 0) = \frac{\xi 1}{\lambda\sqrt{M^2-1}}\frac{1}{p}\frac{p}{p+\nu}\frac{p}{\sqrt{p^2-1}}T(\sqrt{p^2-1},\ y_1,\ 0) \qquad (2.21)$$

The function $\psi(x_1,\ y_1,\ 0)$ can be constructed by proceeding from the relation (4)

$$\frac{p}{\sqrt{p^2-1}}\Phi(\sqrt{p^2-1}) \doteqdot f(t) + \int_0^t f(\sqrt{t^2-\tau^2})\, I_1(\tau)\, d\tau \qquad (2.22)$$

Here $I_1(r)$ is the first order Bessel function of imaginary argument.

Using the inversion theorem and relation (2.22), we obtain for $\psi(x_1,\ y_1,\ 0)$ the following:

$$\psi(x_1,\ y_1,\ 0) = \frac{1}{\lambda\sqrt{M^2-1}}\int_0^{x_1} e^{-\nu(x_1-\xi_1)}\left\{\left[\frac{\partial\psi_0^*(\xi_1,\ y_1,\ z_1)}{\partial\xi_1}\right]_{z_1=0} + \right.$$

$$\left. + \int_0^{\xi_1}\left[\frac{\partial\psi_0^*(\sqrt{\xi_1^2-\xi_2^2},\ y_1,\ z_1)}{\partial\xi_2}\right]_{z_1=0} I_1(\xi_2)\, d\xi_2\right\} d\xi_1 \qquad (2.23)$$

Introducing a new variable $\sigma_1 = \sqrt{(\xi_1^2 - \xi_2^2)}$ and applying the well known relation $dI_0(z)/dz = I_1(z)$ to the function $I_1\sqrt{(\xi_1^2 - \sigma_1^2)}$, we write the inner integral in (2,23) in the form

$$-\int_0^{\xi_1}\left[\frac{\partial\psi_0^*(\sigma_1,\ y_1,\ z_1)}{\partial\sigma_1}\right]_{z_1=0}\frac{d}{d\sigma_1}\left[I_0(\sqrt{\xi_1^2-\sigma_1^2})\right] d\sigma_1$$

Then we obtain for $\psi(x_1,\ y_1,\ 0)$ the form

$$\psi(x_1,\ y_1,\ 0) = \frac{1}{\lambda\sqrt{M^2-1}}\int_0^{x_1} e^{-\nu(x_1-\xi_1)}\left\{\left[\frac{\partial\psi_0^*(\xi_1,\ y_1,\ z_1)}{\partial\xi_1}\right]_{z_1=0} - \right.$$

$$\left. - \int_0^{\xi_1}\left[\frac{\partial\psi_0^*(\sigma_1,\ y_1,\ z_1)}{\partial\sigma_1}\right]_{z_1=0}\frac{d}{d\sigma_1}\left[I_0(\sqrt{\xi_1^2-\sigma_1^2})\right] d\sigma_1\right\} d\xi_1 \qquad (2.24)$$

Here

$$\left[\frac{\partial\psi_0^*(\xi_1,\ y_1,\ z_1)}{\partial\xi_1}\right]_{z_1=0} = \begin{cases} -\pi^{-1}\arccos(1-2y_1/\xi_1) & \text{for } \xi_1 > y_1 \\ -1 & \text{for } \xi_1 < y_1 \end{cases} \qquad (2.25)$$

In the variables x, y, z,

$$\psi(x,\ y,\ 0) = \frac{1}{\lambda\sqrt{M^2-1}}\int_0^{x} e^{-\beta(x-\xi)}\left\{\left[\frac{\partial\psi_0^*(\lambda\xi,\ \lambda\sqrt{M^2-1}y,\ \lambda\sqrt{M^2-1}z)}{\partial\xi}\right]_{z=0} - \right.$$

$$-\int_0^{\xi}\left[\frac{\partial\psi_0^*(\lambda\sigma,\ \lambda\sqrt{M^2-1}y,\ \lambda\sqrt{M^2-1}z)}{\partial\sigma}\right]_{z=0}\frac{d}{d\sigma}[I_0(\lambda\sqrt{\xi^2-\sigma^2})]\,d\sigma\Big\}\,d\xi \tag{2.26}$$

Here

$$\left[\frac{\partial\psi_0^*(\lambda\xi,\ \lambda\sqrt{M^2-1}y,\ \lambda\sqrt{M^2-1}z)}{\partial\xi}\right]_{z=0}=$$

$$=\begin{cases}-\lambda\pi^{-1}\arccos(1-2(y/\xi))\sqrt{M^2-1}, & \text{for } \xi>\sqrt{M^2-1}y\\ -\lambda & \text{for } \xi<\sqrt{M^2-1}y\end{cases} \tag{2.27}$$

Finally, using (2.3), the perturbation velocity potential ϕ for points on the wing surface may be represented as follows:

$$\varphi(x,\ y,\ 0,\ t)=\frac{e^{\alpha t}}{\lambda\sqrt{M^2-1}}\int_0^{x}e^{+\beta\xi}\Big\{\left[\frac{\partial\psi_0^*(\lambda\xi,\ \lambda\sqrt{M^2-1}\,y,\ \lambda\sqrt{M^2-1}\,z)}{\partial\xi}\right]_{z=0}-$$

$$-\int_0^{\xi}\left[\frac{\partial\psi_0^*(\lambda\sigma,\ \lambda\sqrt{M^2-1}\,y,\ \lambda\sqrt{M^2-1}\,z)}{\partial\sigma}\right]_{z=0}\frac{d}{d\sigma}[I_0(\lambda\sqrt{\xi^2-\sigma^2})]\,d\sigma\Big\}\,d\xi \tag{2.28}$$

Here the partial derivatives in square brackets are given by expression (2.27).

The pressure acting on the wing, according to (1.5) and (2.7), will be

$$p(x_1,\ y_1,\ 0,\ t)=p^+-p^-=2\rho_\infty U\lambda e^{\alpha t+\nu x_1}\left[\left(\nu+\frac{\alpha}{U\lambda}\right)\psi(x_1,\ y_1,\ 0)+\frac{\partial\psi(x_1,\ y_1,\ 0)}{dx_1}\right] \tag{2.29}$$

Using (2.29) it is possible to obtain an expression for the lift P of the wing. Let l be the wing span, in the y direction, and h the dimension in the x direction. Then

$$P=\int_0^{l}\int_0^{h}p(x,\ y,\ 0,\ t)\,dx\,dy=\frac{1}{\lambda^2\mu}\int_0^{\lambda\mu l}\int_0^{\lambda h}p(x_1,\ y_1,\ 0,\ t)\,dx_1dy_1=$$

$$=\frac{2\rho_\infty U}{\lambda\mu}e^{\alpha t}\int_0^{\lambda\mu l}\int_0^{\lambda h}\left[e^{\nu x_1}\left(\nu+\frac{\alpha}{U\lambda}\right)\psi(x_1,\ y_1,\ 0)+e^{\nu x_1}\frac{\partial\psi(x_1,\ y_1,\ 0)}{\partial x_1}\right]dx_1dy_1$$

$$(\mu=\sqrt{M^2-1})$$

After some not too difficult transformations, we obtain

$$P = \frac{2\rho_\infty \alpha}{\lambda^2\mu} e^{\alpha t} \int_0^{\lambda\mu l}\int_0^{\lambda h} e^{\nu x_1}\psi(x_1, y_1, 0)\,dx_1dy_1 +$$

$$+ \frac{2\rho_\infty U}{\lambda\mu} e^{\alpha t} \int_0^{\lambda\mu l}\int_0^{\lambda h} \frac{\partial}{\partial x_1}[e^{\nu x_1}\psi(x_1, y_1, 0)]\,dx_1dy_1$$

or, putting in expressions (2.24),

$$P = \frac{2\rho_\infty \alpha}{\lambda^3\mu^2} e^{\alpha t} \int_0^{\lambda h}\int_0^{x_1} e^{\nu\xi_1} \int_0^{\lambda\mu l} \left\{\left[\frac{\partial\psi_0^*(\xi_1, y_1, z_1)}{\partial\xi_1}\right]_{z_1=0} -\right.$$

$$\left. - \int_0^{\xi_1}\left[\frac{\partial\psi_0^*(\sigma_1, y_1, z_1)}{\partial\sigma_1}\right]_{z_1=0} \frac{d}{d\sigma_1}[I_0(\sqrt{\xi_1^2-\sigma_1^2})]\,d\sigma_1\right\} dy_1 d\xi_1 dx_1 +$$

$$+ \frac{2\rho_\infty U}{\lambda^2\mu^2} e^{\alpha t} \int_0^{\lambda h}\frac{\partial}{\partial x_1}\int_0^{x_1} e^{\nu\xi_1} \int_0^{\lambda\mu l} \left\{\left[\frac{\partial\psi_0^*(\xi_1, y_1, z_1)}{\partial\xi_1}\right]_{z_1=0} -\right.$$

$$\left. - \int_0^{\xi_1}\left[\frac{\partial\psi_0^*(\sigma_1, y_1, z_1)}{\partial\sigma_1}\right]_{z_1=0} \frac{d}{d\sigma_1}[I_0(\sqrt{\xi_1^2-\sigma_1^2})]\,d\sigma_1\right\} dy_1 d\xi_1 dx_1 \qquad (2.30)$$

Working out the inner integral gives

$$K_1 = \int_0^{\lambda\mu l}\left\{\left[\frac{\partial\psi_0^*(\xi_1, y_1, z_1)}{\partial\xi_1}\right]_{z_1=0} - \int_0^{\xi_1}\left[\frac{\partial\psi_0^*(\sigma_1, y_1, z_1)}{\partial\sigma_1}\right]_{z_1=0} \times\right.$$

$$\left.\times \frac{d}{d\sigma_1}[I_0(\sqrt{\xi_1^2-\sigma_1^2})]d\sigma_1\right\} dy_1 = \int_0^{\lambda\mu l}\left[\frac{\partial\psi_0^*(\xi_1, y_1, z_1)}{\partial\xi_1}\right]_{z_1=0} dy_1 -$$

$$- \int_0^{\xi_1}\left\{\frac{d}{d\sigma_1}[I_0(\sqrt{\xi_1^2-\sigma_1^2})] \int_0^{\lambda\mu l}\left[\frac{\partial\psi_0^*(\sigma_1, y_1, z_1)}{\partial\sigma_1}\right]_{z_1=0} dy_1\right\} d\sigma_1$$

Using (2.25) it is easy to obtain

$$\int_0^{\lambda\mu l}\left[\frac{\partial\psi_0^*(\xi_1, y_1, z_1)}{\partial\xi_1}\right]_{z_1=0} dy_1 = -\frac{2}{\pi}\int_0^{\xi_1}\arccos\left(1 - 2\frac{y_1}{\xi_1}\right)dy_1 -$$

$$- \int_{\xi_1}^{\lambda\mu l-\xi_1} dy_1 = -\lambda\mu l + \xi_1$$

Then

$$K_1 = -\lambda\mu l + \xi_1 + \lambda\mu l \int_0^{\xi_1} \frac{d}{d\sigma_1}[I_0(\sqrt{\xi_1^2 - \sigma_1^2})]\, d\sigma_1 -$$

$$- \int_0^{\xi_1} \sigma_1 \frac{d}{d\sigma_1}[I_0(\sqrt{\xi_1^2 - \sigma_1^2})]\, d\sigma_1 = -\lambda\mu l I_0(\xi_1) +$$

$$+ \int_0^{\xi_1} I_0(\sqrt{\xi_1^2 - \sigma_1^2})\, d\sigma_1 = -\lambda\mu l I_0(\xi_1) + \operatorname{sh} \xi_1$$

Putting this result for K_1 into (2.30) and changing to the variables (x, y, z, t) we obtain the expression for the lift (in absolute magnitude)

$$P = \frac{2\rho_\infty (U + \alpha h)\, l}{\sqrt{M^2 - 1}} e^{\alpha t} \int_0^h e^{\beta x} I_0(\lambda x)\, dx - \frac{2\rho_\infty \alpha l}{\sqrt{M^2 - 1}} e^{\alpha t} \int_0^h x e^{\beta x} I_0(\lambda x)\, dx -$$

$$- 2\rho_\infty a (U + \alpha h) \frac{e^{\alpha t}}{\alpha} \int_0^h e^{\beta x} \operatorname{sh} \lambda x dx + 2\rho_\infty a e^{\alpha t} \int_0^h x e^{\beta x} \operatorname{sh} \lambda x dx \qquad (2.31)$$

3. In the case of a wing of infinite span, whose angle of attack changes exponentially with time, the perturbation velocity potential to be found is $\phi(x, z, t)$; evidently it is necessary to determine a function $\psi(x_1, z_1)$ satisfying the equation

$$\frac{\partial^2 \psi}{\partial z_1^2} - \frac{\partial^2 \psi}{\partial x_1^2} + \psi = 0 \qquad (3.1)$$

and the following conditions. On the profile $[0, h_1]$

$$\left[\frac{\partial \psi}{\partial z_1}\right]_{z_1 = 0} = \frac{1}{\lambda \sqrt{M^2 - 1}} e^{-\nu x_1} \qquad (3.2)$$

In the region where the fluid is not disturbed, i.e. upstream of the leading edge $x_1 = 0$, the function ψ and its derivatives are equal to zero

$$\psi = 0 \qquad (3.3)$$

Equation (3.1) is in the form of the telegraph equation, well-known in mathematical physics, and easily solved by means of Riemann functions. But for consistency of presentation, we shall use operational methods to solve (3.1). Let

$$F(p, z_1) = p \int_0^\infty e^{-p x_1} \psi(x_1, z_1)\, dx_1 \qquad (3.4)$$

be the transform of $\psi(x_1, z_1)$; then the function $F(p, z_1)$ may be determined from an ordinary differential equation

$$\frac{d^2F}{dz_1^2} - (p^2 - 1)F = 0 \tag{3.5}$$

whose general solution has the form

$$F(p, z_1) = Ae^{-\sqrt{p^2-1}z_1} + Be^{\sqrt{p^2-1}z_1} \tag{3.6}$$

Here A and B are constants, which depend on the parameter p.

Using conditions (3.2) and (3.3), we find

$$A = -\frac{1}{\lambda\sqrt{M^2-1}}\frac{p}{p+\nu}\frac{1}{\sqrt{p^2-1}}, \qquad B = 0 \tag{3.7}$$

Thus we obtain for $F(p, z_1)$ the expression

$$F(p, z_1) = -\frac{1}{\lambda\sqrt{M^2-1}}\frac{p}{p+\nu}\frac{1}{\sqrt{p^2-1}}e^{-\sqrt{p^2-1}z_1} \tag{3.8}$$

The following relation is known (4):

$$\frac{p}{\sqrt{(p+a)(p+b)}}e^{-\tau\sqrt{(p+a)(p+b)}} \;\dot{\rightarrow}\; \begin{cases} 0 & \text{for } t < \tau \\ \exp\left(-\frac{a+b}{2}t\right)I_0\left(\frac{a-b}{2}\sqrt{t^2-\tau^2}\right) & \text{for } t > \tau \end{cases} \tag{3.9}$$

From the inversion theorem, and using relation (3.9), we determine the function $\psi(x_1, z_1)$ for points on the profile, i.e., for $z_1 = 0$

$$\psi(x_1, 0_1) = -\frac{1}{\lambda\sqrt{M^2-1}}\int_0^{x_1} e^{-\nu(x_1-\xi_1)}I_0(\xi_1)\,d\xi_1$$

Changing back to the variables x, z, and in view of (2.3), we obtain for the perturbation velocity potential on points of the wing, the following expression:

$$\varphi(x, 0, t) = -\frac{e^{\alpha t}}{\sqrt{M^2-1}}\int_0^x e^{\beta\xi}I_0(\lambda\xi)\,d\xi \tag{3.10}$$

It is easy to see that exactly the same expression may be obtained from equation (2.28) for the perturbation velocity potential of a wing of finite span, by putting

$$\left[\frac{\partial\psi_0^*(\lambda\xi, \lambda\sqrt{M^2-1}\,y, \lambda\sqrt{M^2-1}z}{\partial\xi}\right]_{z=0} = -\lambda$$

The pressure on the wing of infinite span, and angle of attack varying exponentially with time, is

$$p(x,\ 0,\ t) = p^+ - p^- = 2\rho_\infty \left[\frac{\varphi(x,\ 0,\ t)}{\partial t} + U\frac{\partial\varphi(x,\ 0,\ t)}{\partial x}\right] =$$

$$= 2\rho_\infty \left[-\frac{\alpha}{\sqrt{M^2-1}} e^{\alpha t} \int_0^x e^{\beta\xi} I_0(\lambda\xi)\, d\xi - \frac{U}{\sqrt{M^2-1}} e^{\alpha t} \frac{\partial}{\partial x} \int_0^x e^{\beta\xi} I_0(\lambda\xi)\, d\xi\right] \qquad (3.11)$$

The lift of a wing section with chord h is

$$P = \frac{2\rho_\infty(U+\alpha h)}{\sqrt{M^2-1}} e^{\alpha t} \int_0^h e^{\beta x} I_0(\lambda x)\, dx - \frac{2\rho_\infty \alpha}{\sqrt{M^2-1}} e^{\alpha t} \int_0^h x e^{\beta x} I_0(\lambda x)\, dx \qquad (3.12)$$

This is in complete agreement with expression (2.31) found earlier for the lift of a wing of finite span, from which (3.12) may be obtained by going over to the limit $l \to \infty$.

4. We return to the problem of determining the perturbation velocity potential $\phi(x,\ y,\ z,\ t)$ as a solution of equation (1.1) with boundary conditions (1.2)-(1.4), i.e. to the case for which the normal velocity component on the surface of a rectangular wing changes with time in an arbitrary manner:

$$\left[\frac{\partial\varphi}{\partial z}\right]_{z=0} = f(t) \quad (-\infty \leqslant t \leqslant 0) \qquad (4.1)$$

A case of interest is a gust having the following form:

$$\left[\frac{\partial\varphi}{\partial z}\right]_{z=0} = f(t) = \begin{cases} 0 & \text{for } -\infty \leqslant t \leqslant -t_0 \\ 1 & \text{for } -t_0 \leqslant t \leqslant 0 \end{cases} \qquad (4.2)$$

To determine the perturbation velocity potential $\phi(x,\ y,\ z,\ t)$ we shall make use of the solution already obtained (2.28).

We expand the function $f(t)$ in powers of e^t (in certain cases it is convenient to make the expansion in terms of e^{at}, where a is an arbitrary parameter)

$$f(t) = \lim_{N\to\infty} \sum_{r=0}^{N} a_r e^{rt} \qquad (-\infty \leqslant t \leqslant 0) \qquad (4.3)$$

We shall look for a solution of equation (1.1) in the form

$$\varphi(x,\ y,\ z,\ t) = \lim_{N\to\infty} \sum_{r=0}^{N} a_r \varphi_r(x,\ y,\ z,\ t) \qquad (4.4)$$

where $\phi_r(x,\ y,\ z,\ t)$ is the particular solution (2.28) of equation (1.1) which was obtained earlier.

We introduce a change of variable. Let $\tau = e^t$. Then

$$f(t) = f_1(\tau) = \lim_{N\to\infty} \sum_{r=0}^{N} a_r \tau^r \qquad (0 \leqslant \tau \leqslant 1) \tag{4.5}$$

Then put $\tau = 1/2(q + 1)$, or $q = 2\tau - 1$. It can be seen that for $f_1(t)$ we obtain

$$f_1(\tau) = g(q) = \lim_{N\to\infty} \sum_{r=0}^{N} a_r' q^r \qquad (-1 \leqslant q \leqslant 1) \tag{4.6}$$

where $g(q)$ is function which is given on the interval $[-1, +1]$, on which it has a finite number of points of discontinuity of first order; it satisfies the conditions for an expansion in series of Legendre polynomials [5]. We represent $g(q)$ in a series of normalized Legendre polynomials:

$$g(q) = \lim_{N\to\infty} \sum_{n=0}^{N} (n + \tfrac{1}{2})^{1/2} B_n P_n(q) \tag{4.7}$$

Here

$$P_n(q) = \frac{1}{2^n} \sum_{k=0}^{E(n/2)} \frac{(-1)^k (2n-2k)!}{k!\,(n-k)!\,(n-2k)!} q^{n-2k} \tag{4.8}$$

is the orthogonal Legendre polynomial of nth order. The coefficients of the expansion are

$$B_n = \int_{-1}^{+1} (n + \tfrac{1}{2})^{\frac{1}{2}} P_n(q)\, g(q)\, dq \tag{4.9}$$

It may be seen that the series (4.7) for the function $g(q)$ may be written as follows:

$$g(q) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n P_n(q), \qquad P_n(q) = \sum_{k=0}^{n} b_{nk} q^k \tag{4.10}$$

To determine the coefficients b_{nk} we shall make some transformations in the well known representation (4.8) of the Legendre polynomials. We put $2k = m$; then

$$P_n(q) = \frac{1}{2^n} \sum_{m=0}^{n} (-1)^{\frac{m}{2}} \frac{(-1)^m + 1}{2} \frac{\Gamma(2n-m+1)}{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(n-\frac{m}{2}+1\right)\Gamma(n-m+1)} q^{n-m} \tag{4.11}$$

Now let $n - m = k$ in expression (4.11). We obtain a representation of the

Legendre polynomial of nth order in the form of a series in increasing powers of q,

$$P_n(q) = \frac{1}{2^n}\sum_{k=0}^{n}(-1)^{\frac{n-k}{2}}\frac{(-1)^{n-k}+1}{2}\frac{\Gamma(n+k+1)}{F\left(\frac{n-k}{2}+1\right)\Gamma\left(\frac{n+k}{2}+1\right)\Gamma(k+1)}q^k \tag{4.12}$$

Equating (4.10) and (4.12), we find

$$b_{nk} = \frac{1}{2^n}(-1)^{\frac{n-k}{2}}\frac{(-1)^{n-k}+1}{2}\frac{\Gamma(n+k+1)}{\Gamma\left(\frac{n-k}{2}+1\right)\Gamma\left(\frac{n+k}{2}+1\right)\Gamma(k+1)}$$

Thus

$$g(q) = \lim_{N\to\infty}\sum_{n=0}^{N} a_n P_n(q) = \lim_{N\to\infty}\sum_{n=0}^{N} a_n \sum_{k=0}^{n} b_{nk}q^k = \lim_{N\to\infty}\sum_{l=0}^{N}\sum_{k=l}^{N} a_k b_{kl} q^l,$$

Here

$$a_k = \left(k+\frac{1}{2}\right)\int_{-1}^{+1} P_k(q) g(q) dq$$

$$b_{kl} = \frac{1}{2^k}(-1)^{\frac{k-l}{2}}\frac{(-1)^{k-l}+1}{2}\frac{\Gamma(k+l+1)}{\Gamma\left(\frac{k-l}{2}+1\right)\Gamma\left(\frac{k+l}{2}+1\right)\Gamma(l+1)} \tag{4.13}$$

Finally, the function $g(q)$ may be expressed as follows:

$$g(q) = \lim_{N\to\infty}\sum_{l=0}^{N} a_l' q^l \tag{4.14}$$

where

$$a_l' = \sum_{k=l}^{N}\left[\left(k+\frac{1}{2}\right)\frac{1}{2^k}\int_{-1}^{+1} P_k(q) g(q) dq\right](-1)^{\frac{k-l}{2}}\frac{(-1)^{k-l}+1}{2}\times$$
$$\times\frac{\Gamma(k+l+1)}{\Gamma\left(\frac{k-l}{2}+1\right)\Gamma\left(\frac{k+l}{2}+1\right)\Gamma(l+1)} \tag{4.15}$$

Let us return to the variable τ. It is clear that

$$q^l = (2\tau-1)^l = \sum_{p=0}^{l}\frac{l!}{p!(l-p)!}(-1)^{l-p}(2\tau)^p$$

Then we can write

$$g(q) = \lim_{N\to\infty} \sum_{l=0}^{N} a_l' q^l = \lim_{N\to\infty} \sum_{l=0}^{N} a_l' \sum_{p=0}^{l} \frac{(-1)^{l-p} l!}{p!\,(l-p)!} (2\tau)^p =$$

$$= \lim_{N\to\infty} \sum_{l=0}^{N} a_l' \sum_{p=0}^{l} a''_{lp} \tau^p = \lim_{N\to\infty} \sum_{r=0}^{N} \sum_{s=r}^{N} a_s' a''_{sr} \tau^r$$

where

$$a''_{sr} = \frac{(-1)^{s-r} s!\, 2^r}{r!\,(s-r)!}$$

Therefore, in view of (4.6),

$$f_1(\tau) = \lim_{N\to\infty} \sum_{r=0}^{N} \left\{ \sum_{s=r}^{N} \left[\sum_{k=s}^{N} \left(k+\tfrac{1}{2}\right) \frac{1}{2^k} \int_{-1}^{+1} P_k(q)\, g(q)\, dq\, (-1)^{\frac{k-s}{2}} \frac{(-1)^{k-s}+1}{2} \times \right.\right.$$
$$\left.\left. \times \frac{\Gamma(k+s+1)}{\Gamma\left(\frac{k-s}{2}+1\right)\Gamma\left(\frac{k+s}{2}+1\right)\Gamma(s+1)} \right] \frac{(-1)^{s-r}\Gamma(s+1)}{\Gamma(s-r+1)} \right\} \frac{2^r}{\Gamma(r+1)} \tau^r \tag{4.16}$$

and finally, for $f(t) = \left[\frac{\partial \varphi}{\partial z}\right]_{z=0}$, we obtain in accordance with (4.5)

$$f(t) = \lim_{N\to\infty} \sum_{r=0}^{N} a_r e^{rt} \tag{4.17}$$

where

$$a_r = \sum_{s=r}^{N} \left[\sum_{k=s}^{N} \left(k+\frac{1}{2}\right) \frac{1}{2^k} \int_{-1}^{+1} P_k(q)\, g(q)\, dq\, (-1)^{\frac{k-s}{2}} \frac{(-1)^{k-s}+1}{2} \times \right.$$
$$\left. \times \frac{\Gamma(k+s+1)}{\Gamma\left(\frac{k-s}{2}+1\right)\Gamma\left(\frac{k+s}{2}+1\right)\Gamma(s+1)} \right] \frac{(-1)^{s-r}\Gamma(s+1)2^r}{\Gamma(s-r+1)\,\Gamma(r+1)} \tag{4.18}$$

Using (4.4), the perturbation velocity potential is

$$\varphi(x, y, 0, t) = \tag{4.19}$$
$$= \frac{1}{\sqrt{M^2-1}} \lim_{N\to\infty} \sum_{r=0}^{N} a_r \frac{e^{rt}}{\lambda_1 r} \int_0^x e^{\beta_1 r \xi} \left\{ \left[\frac{\partial \psi_0^*(\lambda_1 r\xi, \lambda_1 r\sqrt{M^2-1}\, y, \lambda_1 r \sqrt{M^2-1}\, z)}{\partial \xi} \right]_{z=0} - \right.$$
$$\left. - \int_0^{\xi} \left[\frac{\partial \psi_0^*(\lambda_1 r\sigma, \lambda_1 r\sqrt{M^2-1}\, y, \lambda_1 r\sqrt{M^2-1}\, r)}{\partial \sigma} \right]_{z=0} \frac{d}{d\sigma} [I_0(\lambda_1 r \sqrt{\xi^2-\sigma^2}]\, d\sigma \right\} d\xi$$

Here

$$\beta_1 = -\frac{M}{a} \frac{1}{M^2-1}, \qquad \lambda_1 = \frac{1}{a} \frac{1}{M^2-1}$$

The lift of a wing with finite span l and chord h is

$$P = \frac{2\rho_\infty}{\sqrt{M^2-1}} \lim_{N\to\infty} \sum_{r=0}^{N} a_r \Big[l(U+rh)e^{rt}\int_0^h e^{\beta_1 rx} I_0(\lambda_1 rx)\,dx -$$

$$- lre^{rt}\int_0^h xe^{\beta_1 rx} I_0(\lambda_1\ rx)\,dx - \sqrt{M^2-1}\,a(U+rh)\frac{e^{rt}}{r}\int_0^h e^{\beta_1 rx}\operatorname{sh}\lambda_1 rx\,dx +$$

$$+ \sqrt{M^2-1}\,ae^{rt}\int_0^h xe^{\beta_1 rx}\operatorname{sh}\lambda_1 rx dx\Big] \tag{4.20}$$

For a wing section (case of a wing of infinite span),

$$\varphi(x,0,t) = -\frac{1}{\sqrt{M^2-1}} \lim_{N\to\infty} \sum_{r=0}^{N} a_r e^{rt}\int_0^x e^{\beta_1 r\xi} I_0(\lambda_1 r\xi)\,d\xi \tag{4.21}$$

$$P = \frac{2\rho_\infty}{\sqrt{M^2-1}} \lim_{N\to\infty} \sum_{r=0}^{N} a_r \left[(U+rh)e^{rt}\int_0^h e^{\beta_1 rx} I_0(\lambda_1 rx)\,dx - re^{rt}\int_0^h xe^{\beta_1 rx} I_0(\lambda_1 rx)\,dx\right] \tag{4.22}$$

It must be noted that with increasing r the coefficients a_r of some functions will tend to infinitely large values. However any function may be approximated by means of a finite number of Legendre polynomials to any degree of accuracy. The derivation of aerodynamic characteristics in such cases is to be regarded as a certain asymptotic process.

In the case of the gust (4.2), the function $f(t)$ is discontinuous and, therefore, the coefficients $a_r \to \infty$ as $r \to \infty$. In this case the function $f(t)$ may be approximately represented in the form

$$f(t) = \sum_{r=0}^{N} a_r e^{xrt}$$

A satisfactory approximation is obtained for $N \approx 15$. From (4.19) and (4.20), approximate values of the aerodynamic characteristics are obtained.

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